KSU-GRG-97-12

Quantum mechanics in curved space and quantization of polynomial Hamiltonians[1]

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July 20, 1997

Abstract

The quantization of a single particle without spin in an appropriate curved space-time is considered. The Hamilton formalism on reduced space for a particle in a curved space-time is constructed and the main aspects of quantization scheme are developed. These investigations are applied to quantization of the particle Hamiltonian in an appropriate curved space-time. As an example the energy eigenvalues in Einstein universe are calculated. In the last sections o the paper approximation for small values of momenta of the results previously obtained is considered as well as quantization of polynomial Hamiltonians of general type is discussed.

[1] The talk presented at the XVI*th* Workshop on Geometric Methods in Physics. June 30 - July 6, 1997, Bialowieza, Poland.

1 Introduction

One of the first attempts to construe the quantum mechanics of a relativistic particle was undertaken by P. A. M. Dirac [7] who have considered various forms of relativistic dynamics corresponding to different types of 3-surfaces in Minkowski space-time. Dirac considered only the case of the flat space as far as he supposed that the gravitational effects are insignificant on the quantum scales. Further development of quantum gravity has resulted in consideration of quantum systems in which the effects of general relativity are sufficiently strong.

However, up to the present time doesn't exist the constructive quantum theory in curved space-time. The difficulties in development of such theory are connected to the problem of construction the Hamilton formalism for mechanical systems in curved space-time. They appear because we have a constrained theory of parametrized type, i.e. each solution curve of which lies within an phase orbit. Note that the Einstein's general relativity is parametrized theory too [11]). Therefore, we can hope that the quantization of particle dynamics in curved space-time could answer questions arising in quantum gravity.

At the present paper we consider quantization of a single particle without spin in an appropriate curved space-time. There are two approaches to the quantization of constrained systems (as well as to the curved space quantization). The first approach corresponds to the resolution of the constraints at the quantum level and the second approach is based on the symmetry reduction procedure [16] for constrained systems (see [8, 17] for discussion on the problem of commutation of reduction and quantization).

Which approach more suitable to the case of quantum particle in curved space? First approach has the disadvantage that in general case it is a complicated problem to solve the quantum constraints. However, in certain cases it is possible to do it by the use of $3 + 1$-decomposition [21].

We propose below a way of quantization which use mostly the second approach, i.e. solving the constraints at classical level. It has also one defect that the quantum theory will not be generally covariant because the Hamiltonian and the Poisson brackets are not covariant expressions. However, it seems in the case, that using of reduced phase space method makes physical situation more clear and calculations more simple.

One more problem we discuss here is the quantization of functions which are polynomial in momenta (we call them polynomial Hamiltonians). The example of general relativistic particle (and quantum gravity) shows the importance of this problem. The approach which allows to quantize polynomial Hamiltonians on smooth manifolds is developed in the paper.

The plan of the paper is as follows. In Sec. 2 the Hamilton formalism on reduced space for particle in curved space-time is described. In the next section the main aspects of quantization scheme (based on geometric quantization approach) are developed. Here we also describe the existing methods of curved space quantization (for more complete review see [9, 11, 20, 21]) as well as problems these methods faced to. Sec. 4 is devoted to quantization of general relativistic Hamiltonian in an appropriate curved

space-time. As example the energy eigenvalues of the particle in Einstein universe are calculated in this section. In Sec. 5 we consider the approximation of the results previously obtained for small values of momenta. In the last section of the paper quantization of polynomial Hamiltonians is discussed.

2 Hamilton formulation of particle dynamics in curved space

We shall not discuss in this paper the aspects of physical definition of particle in curved space [4]. We proceed from the naive definition of particle as an object which configuration space is a pseudo Riemannian manifold $(\mathcal{M}, g)$ — the space-time manifold. Our goal is to develop the Hamilton formulation for this system.

Let us consider two parts of the Hamilton formulation which are: the phase space reduction and the reduction at the level of action. First part results to physical phase space and corresponds to solving of constraints. The second part should lead to correct general relativistic Hamiltonian. We consider both these parts in order to display different features of the system.

To start the discussion we first investigate the phase space reduction. The phase space of our system is cotangent bundle $N = T^*\mathcal{M}$ over space-time manifold $\mathcal{M}$ with symplectic 2-form ω. Let $\pi : T^*\mathcal{M} \to \mathcal{M}$ be the canonical projection. In a local chart $(\pi^{-1}(U), x^i, p_i)$, $i = 0, \ldots, 3$, $U \subseteq \mathcal{M}$ the 2-form ω is given by the formula (the *canonical symplectic form*)

$$\omega = \sum dp_i \wedge dx^i. \tag{1}$$

The dynamics of the particle is defined by constraints and Hamiltonian. The only constraint is the mass-shell condition $C = 1/2(g^*(p,p) - m^2) = 0$, $p \in T_x^*\mathcal{M}$ (here g^* is induced bilinear form on $T_x^*\mathcal{M}$). The problem of finding physical Hamiltonian of the theory will be considered later. Now we only note that we can choose Hamiltonian in the form $H_{\mathcal{M}} = C = 1/2(g^*(p,p) - m^2)$. It is constant on the constraint surface $\Sigma = \{(x,p) \in N \,|\, C = 0\}$ and, hence, phase trajectories in N preserve Σ. It means that the considered system is of parametrized type.

Let $\mathrm{ad}\,(f)$ be Hamiltonian vector field of the function $f \in C^\infty(N)$ defined by the condition

$$\iota(\mathrm{ad}\,(f))\omega = -df,$$

where ι is the internal product. In local coordinates this formula reads as

$$\mathrm{ad}\,(f) = \omega^{ij}\partial_j f \partial_i.$$

Vector field $\mathrm{ad}\,(C)$ defines one-parameter transformation group ϕ_C^τ of symplectomorphisms in N defined by the equations

$$\frac{d\phi_C^\tau x}{d\tau} = \mathrm{ad}\,(C). \tag{2}$$

The Lie group ϕ^τ_C acts on symplectic manifold (N,ω) by symplectomorphisms and preserves the constraint surface Σ. Reduction procedure corresponds to the factorization of Σ by this action and to the transition to an orbit space $\mathcal{O}$ (physical phase space). This space is a symplectic manifold if the following conditions are obeyed [1, 16]

1) Σ is a smooth manifold,
2) ϕ^τ_C acts on Σ without fixed points,
3) the action of group ϕ^τ_C is *proper*, i.e for mapping $(\phi^\tau_C, x) \mapsto (\phi^\tau_C(x), x)$ images of compact sets are compact.

The first condition is obviously satisfied in the considered case. If 1) and 2) holds then the condition 3) is satisfied also. In order to prove this fact we note that since the group ϕ^τ_C acts on the constraint surface Σ without fixed points then any orbit γ of this action can be identified with the group ϕ^τ_C. By choosing a suitable reparametrization $\tau' = \tau'(\tau)$ one can derive that the action of group ϕ^τ_C on γ corresponds to the left-shift action of the group on itself. Because left-shift action is proper, the condition 3) is satisfied.

Let us pass to the second condition. It is satisfied if and only if the vector field $\mathrm{ad}\,(C)$ is nonvanishing on Σ. In a local chart this means

$$g^{\alpha\beta}p_\beta \neq 0 \quad \text{or} \quad \partial_\sigma g^{\alpha\beta}p_\alpha p_\beta \neq 0. \tag{3}$$

If $m \neq 0$ (as it is in the case of massive particle) then $p \neq 0$ and (3) is satisfied. If $m = 0$ (the case of massless particle) then in order to satisfy the second condition one should consider the space $\Sigma \backslash \Sigma_0$ where $\Sigma_0 = \{(x, 0) \in T^*\mathcal{M}\}$, $\Sigma_0 \subset \Sigma$. So we prove that conditions 1) – 3) are satisfied and Hamilton formulation can be obtained by reduction of the initial system.

For physical applications the concrete realization of the reduction procedure, i.e. the orbit space construction have exceptional significance. For this purpose we shall use *transversal surface* [1, 10, 11] $N_{\rm tr}$ which is in our case the cotangent bundle T^*M_0 for some spacelike 3-surface $M_0 \subset \mathcal{M}$. We shall call M_0 *instant* 3*-surface*. Transversal surface is realization of the orbit space, i.e. it is transversal to all orbits and intersects each orbit only in one point. Moreover, $N_{\rm tr}$ is a symplectic manifold with the symplectic form projected from N.

A smooth function $T : \mathcal{M} \to \mathbb{R}$ increasing along any non-spacelike curve is called *global time function* [3]. A Lorentzian manifold $(\mathcal{M}, g)$ admits the global time function if and only if $(\mathcal{M}, g)$ is *stably casual*[2]. In general case there are no natural way of choosing global time function on a stably casual space-time. Let $\mathrm{grad}\,(f)$ be the gradient flow of a smooth function $f : \mathcal{M} \to \mathbb{R}$, i.e. $g(\mathrm{grad}\,(f), X) = df(X)$ for any vector field X on $\mathcal{M}$. If $\mathrm{grad}\,(f)$ is time-like, then f increases or decreases along any non-spacelike curve and f or $-f$ can be taken as global time function.

Let a space-time $(\mathcal{M}, g)$ be stably casual with the global time function T. Then we can take the instant 3-surface in the form $M_0 = \{x \in \mathcal{M} \,|\, T(x) = T_0 = \mathrm{const}\}$.

[2]It means that there are no closed timelike or null curves in any Lorentzian metric which is sufficiently near to the metric g in the sense of some C^0 open topology on the space $Lor(\mathcal{M})$ of all C^2 Lorentz metric on the space-time manifold $\mathcal{M}$ [12].

Let $x^i = x^i(v^\mu)$ are the local equations of embedding for M_0, then we get the reduced phase space T^*M_0 with the canonical symplectic form

$$\omega = \sum dp_\mu \wedge dv^\mu, \qquad \mu = 1, 2, 3.$$

The reduction of phase space doesn't give us the Hamiltonian of the theory. In order to solve this problem we shall consider the action of the system. In the case of scalar particle in curved space $\mathcal{M}$ it can be given by the following formula

$$S[x] = m \int dt \sqrt{g(\dot{x}, \dot{x})}. \tag{4}$$

This action has to be supplemented by the mass-shell condition

$$g(\dot{x}, \dot{x}) = m^2 \tag{5}$$

which is the constraint of the theory. Let us rewrite the action in the form

$$S[x, e] = 1/2 \int d\tau (\frac{g(\dot{x}, \dot{x})}{e} + em^2). \tag{6}$$

It is easy to prove that these two forms of action define equivalent forms of dynamics. Now we can obtain from (6) the primary constraint $p_e = 0$ and the canonical Hamiltonian

$$H_{\text{can}} = eH_{\mathcal{M}}, \qquad H_{\mathcal{M}} = \frac{1}{2}(g^*(p, p) - m^2). \tag{7}$$

The Poisson brackets of the primary constraint and the Hamiltonian gives the secondary constraint

$$\{p_e, H_{\text{can}}\} = H_{\mathcal{M}} = 0. \tag{8}$$

Now we can rewrite the initial action (6) in the Hamiltonian form

$$S = \int_\gamma \Theta \tag{9}$$

where $\Theta = \theta - eH_{\mathcal{M}}dt$ is *Poincarè-Cartan integral invariant* [1] and $\theta = p_0 dT + p_\mu dv^\mu$ is *action* 1-*form* on the phase space N defined by the condition $\omega = d\theta$ up to addition of an exact 1-form df. The integral in (9) should be taken along the graph γ of the system with Hamiltonian $H_{\mathcal{M}}$ in evolution space $N \times \mathbb{R}$.

Using (8), we get $S = \int_{\tilde{\gamma}} \theta|_\Sigma$ where $\tilde{\gamma}$ is the projection of γ on the reduced phse space N_{tr}

Let us choose new coordinate system (x^A, C, φ), $A = 1, \ldots, 6$ so that

$$q^i = q^i(x^A, C, \varphi), \qquad p^i = p^i(x^A, C, \varphi).$$

Then we can write θ in the form

$$\theta = p_i \frac{\partial q^i}{\partial x^A} dx^A + p_i \frac{\partial q^i}{\partial C} dC + + p_i \frac{\partial q^i}{\partial \varphi} d\varphi,$$

hence,

$$\theta|_\Sigma = (\zeta_A \frac{dx^A}{d\varphi} - H_\varphi) d\varphi$$

where

$$\zeta_A = p_i \frac{\partial q^i}{\partial x^A}, \qquad H_\varphi = -p_i \frac{\partial q^i}{\partial \varphi}. \tag{10}$$

Now we can write down the action in new coordinates

$$S = \int \zeta_A \frac{dx^A}{d\varphi} - H_\varphi) d\varphi.$$

We find the Hamiltonian form for action of the system with the Hamiltonian (10). The phase space of this system is N_{tr} whose symplectic form (locally) is

$$\omega = \frac{\partial \zeta_B}{\partial x^A} dx^A \wedge dx^B.$$

Here the function $H(\varphi, x^1, \ldots, x^6)$ is defined by

$$C(-F, H, x^1, \ldots, x^6) = 0.$$

As an example we consider the instant form of particle dynamics in curved space-time $\mathcal{M}$ (which is rather similar to instant dynamics in Minkowski space [7]). Let $\mathcal{M}$ be a stably casual Lorentzian manifold with global time function T and instant 3-surface M_0. Let v^μ are the local coordinates on M_0. If we take (T, v^μ) as coordinate system in $\mathcal{M}$ then $g_{0\mu} = 0$, $g_{00} > 0$ and the functions $\gamma_{\alpha\beta} = -g_{\alpha\beta}/g_{00}$, $g^{\alpha\mu} g_{\mu\beta} = \delta^\alpha_\beta$ are the components of Riemannian metric on M_0. Taking $\varphi = p_0$ we find from (10)

$$H = \pm\sqrt{\gamma^{\mu\nu} p_\mu p_\nu + g_{00}\, m^2}.$$

As the result, we get the following reduced action form

$$S_{\rm red} = \int dT (p \frac{dv}{dT} - H), \quad (v, p) \in N_{\rm tr} = T^* M_0.$$

From here it follows that H is the Hamiltonian of the mechanical system with phase space $N_{\rm tr} = T^* M_0$. This system is equivalent to the initial sytem with Hamiltonian $H_{\mathcal{M}} = 1/2(g^*(p,p) - m^2)$ and the constraint $C = H_{\mathcal{M}} = 0$.

It is possible to construct another forms of dynamics by choosing different types of 3-surfaces (timelike or null) in the reduction procedure. This will result to another choice of the function φ. These forms of dynamics could be useful for systems which don't admit the description in terms of Hamilton mechanics on *instant* 3-surface but *do admit* Hamilton formulation achieved by the use of reduction procedure (in fact for such systems should be chosen another representation for orbit spaces). In the following we shall restrict ourselves only to the consideration of the instant form of dynamics.

3 Geometric quantization

In this section we describe the geometric quantization approach. Later it will be used for quantization of previously considered system. Let us start from formulation of quantization axioms [13, 15, 20, 23].

Quantization is the linear map $\mathcal{Q} : f \mapsto \hat{f}$ of Poisson (sub)algebra $C^\infty(N)$ into the set of operators in some (pre)Hilbert space $\mathcal{H}$ possessing the following properties:

(Q1) $\hat{1} = 1$;

(Q2) $\widehat{\{f, g\}}_h = \frac{i}{h}(\hat{f}\hat{g} - \hat{g}\hat{f})$;

(Q3) $\hat{\bar{f}} = (\hat{f})^*$;

(Q4) for a complete set of functions $f_1, \dots, f_n$ the operators $\hat{f}_1, \dots, \hat{f}_n$ also form a complete set.

Here the bar is the complex conjugation, the star denotes the conjugation of operator and $h = 2\pi\hbar$ is the Planck constant.

Geometric quantization approach was developed [15, 13, 20] to help solve some difficulties of canonical quantization scheme. These difficulties are especially essential in the case of the phase spaces with non-trivial topology [9, 13].

Let (N, ω) be a n-dimensional symplectic manifold. Geometric quantization procedure consists of the following main parts.

a) *Prequantization line bundle* $\mathcal{L}$ that is a Hermitian line bundle over N with connection D and D-invariant[3] Hermitian structure $<,>$. The curvature form Ξ of the prequantization bundle $\mathcal{L}$ have to coincide with the form $h^{-1}\omega$.

b) *Polarization* F that is an involutive Lagrange distribution in $TN \otimes_{\mathbb{R}} \mathbb{C}$.

c) *Metaplectic structure* which consists of the *bundle of metalinear frames in* TN and the *bundle* $\mathcal{L} \otimes \sqrt{\wedge^n F}$ *of* $\mathcal{L}$*-valued half-forms normal to the polarization* F (see, for example, [20]).

If these three structures are defined on (M, ω), then the Hilbert space $\mathcal{H}$ of the system consists of such sections μ of the bundle $\mathcal{L} \otimes \sqrt{\wedge^n F}$ which are covariantly constant with respect to the polarization F. The *Souriau-Kostant prequantization formula*

$$\mathcal{P}(f) = f - i\hbar D_{\mathrm{ad}\,(f)}. \tag{11}$$

provides the set of the operators $\mathcal{P}(f)$ satisfying the first three quantization axioms. In order to satisfy the axiom (Q4) one should introduce the polarizations and use the *Blattner-Costant-Sternberg (BKS) kernel.* The BKS kernel is a bilinear mapping $K : \mathcal{H}_{F_1} \times \mathcal{H}_{F_2} \to \mathbb{R}$ which connects Hilbert spaces for two different polarization F_1 and F_2.

It is possible to offer two approaches for the quantization of a mechanical system. The first approach corresponds to the resolution of the constraints at the quantum level and the second approach is based on the Hamilton formalism for constrained

[3]Recall that D-invariance means that for each pair of sections λ and μ of $\mathcal{L}$ and each real vector field X on M holds $X < \lambda, \mu >=< D_X\lambda, \mu > + < \lambda, D_X\mu >$.

systems reviewed in Sec. 2. What this two approaches give us in application to general relativistic particle case?

If we use the quantum constraints then we have to start from the phase space $(T\mathcal{M}, \omega)$ where ω is given by Eq. (1). Then the operator $\mathcal{Q}(H)$ corresponding to the Hamiltonian (7) allocates admissible (physical) states in the Hilbert space $\mathcal{H}$. If we write the classical constraint in the form $C^2 = 0$, then the quantum constraint is $\mathcal{Q}(C^2) = 0$. Geometric quantization gives the following expression for this operator [6, 20][4]

$$\mathcal{Q}(C^2)\,\psi\,\lambda_0 \otimes \mu_0 = (-\hbar^2(g^{ij}\nabla_i\nabla_j - \frac{1}{6}R) + m^2)\,\psi\,\lambda_0 \otimes \mu_0 \tag{12}$$

where R is the Ricci scalar, $\psi \in C^\infty(M_0)$ and $\lambda_0 \otimes \mu_0$ is a special nonvanishing section of the bundle $\mathcal{L} \otimes \sqrt{\wedge^n F}$ (see the next section). The last formula means that physical states have to obey the conformally-invariant Klein-Gordon-Fock equation [5]. Solutions of this equation form *physical Hilbert space* of our system. In general case it is a complicated problem to solve this quantum constraint. However, in certain cases it is possible to solve this problem by the use of $3+1$ decomposition [21].

At the sane time, solving the constraints at classical level has also the disadvantage that the quantum theory will not be generally covariant in this case because the Hamiltonian and the Poisson brackets are not covariant expressions (such situation reminds the gauge fixing in quantum gauge theories). However, using of reduced phase space makes (in certain cases) physical situation more clear and calculations more simple. It allows to solve problems of quantum mechanics in curved space which would be rather complicated if we would use the first approach.

One more way of solving the constraint at classical level could be proposed. It is based on the fact that the *constraint surface* Σ is a *Poisson manifold* and it is possible to use the method of geometric quantization of Poisson manifolds proposed by I. Vaisman [22] (see also [19]).

4 Quantization in curved space-time

We consider here (see also [14]) geometric quantization of the system investigated in Sec. 2 at classical level. Let $(\mathcal{M}, g)$ be a space-time and M_0 is the instant surface in $\mathcal{M}$ (we note here that M_0 is orientable). Let γ be the Riemannian metric on M_0 projected from $(\mathcal{M}, g)$.

The metric γ can be used to define a global section of the bundle $\sqrt{\wedge^n F}$. We give here short review of this construction, see [20] for complete reference.

First, one can note that there exists isomorphism between the sections μ of $\mathbf{L} \equiv \sqrt{\wedge^n F}$ and the set of complex valued functions $\nu^{\#}$ on the bundle $\mathbf{L}^* \equiv \mathbf{L}\backslash\{0\}$ possessing the property

$$\mu^{\#}(cz) = c^{-1}\mu^{\#}(z), \quad c \in \mathbb{C}.$$

[4] The similar expression for quantization of C^2 was obtained using another approach in [8].

This isomorphism is given by the formula

$$\mu(\pi z) = \mu^{\#}(z)z$$

where π is the projection of the bundle $\mathbf{L}$. If $\{\eta_i\}$ is a metalinear frame [20], corresponding to the polarization F, then we can define a section $\tilde{\mu}$ of $\mathbf{L}$ by the equation $\tilde{\mu}^{\#} \circ \eta = 1$. Let $U \subset N$ be an open domain. We introduce a local section $\mu_0|_U$ of $\mathbf{L}$ over U by

$$\mu_0|_U = \pm|(\det\gamma) \circ \pi|^{1/4}\tilde{\mu}.$$

Using well-known transformation properties of the metric tensor γ and the half-form $\tilde{\mu}$ it can be shown that the sections $\mu_0|_U$ define *global section* μ_0 of $\mathbf{L}$ which doesn't depend on the choice of particular coordinates.

Let us introduce in T^*M_0 *vertical polarization* F spanned by the $\partial/\partial v^\alpha$ where v^α, $\alpha = 1, 2, 3$ are local coordinates in M_0. The Hilbert space $\mathcal{H}$ corresponding to this polarization consists of the sections of the form $\psi(v^\alpha)\lambda \otimes \mu_0$ [20]. The dynamical variables of physical interest are the canonical coordinates v^α, the corresponding momenta p^α and the Hamiltonian H. Coordinates and momenta preserve the polarization and can be easily quantized by the use of Souriau-Kostant prequantization formula (11) which yields

$$\mathcal{Q}(v^\alpha) = v^\alpha \quad \text{and} \quad \mathcal{Q}(p^\alpha) = \frac{i}{\hbar}\frac{\partial}{\partial v^\alpha}.$$

However, the Hamiltonian H doesn't preserve the polarization. Hence, it is necessary to use BKS kernel to quantize H.

As the first example let us consider the problem of finding energy eigenvalues of scalar particle in *Einstein universe*. In this case it is convenient to quantize the square H^2 of the Hamiltonian but not the Hamiltonian itself. The space-time $\mathcal{M}$ in this case is isomorphic to $S^3 \times \mathbb{R}$ with the metric

$$ds^2 = dt^2 - a^2 d\Omega^2$$

where $a \in \mathbb{R}$ and $d\Omega^2 = \sigma_{\mu\nu}dx^\mu dx^\nu$ is the metric on 3-sphere S_1^3 of unit radius. Hamiltonian of the system can be taken in the form

$$H = (a^{-2}\sigma^{\mu\nu}p_\mu p_\nu + m^2)^{1/2}. \tag{13}$$

Similarly with (12) we find

$$\mathcal{Q}(H^2)\psi\ \lambda_0 \otimes \mu_0 = (-\hbar^2(a^{-2}\sigma^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{1}{6}R) + m^2)\psi\ \lambda_0 \otimes \mu_0 \tag{14}$$

where covariant derivatives and Ricci curvature should be calculated for 3-metric $d\Omega^2$. The eigenvalues of the operator $\mathcal{Q}(H^2)$ coincides with squares λ^2 of energy eigenvalues λ. We shall now calculate λ's using the fact that eigenvalues ρ_k of the Laplacian $\triangle$ on 3-sphere S_a^3 of the radius a are [2]

$$\rho_k = -\frac{k(k+2)}{a^2}, \qquad k \in \{0\} \cup \mathbb{N}.$$

The Ricci curvature of S_a^3 is equal to $R = 6a^{-2}$. From here it follows that eigenvalues of the operator $\mathcal{Q}(H^2)$ (which coincide with the stationary energy levels of quantum scalar particle in Einstein universe) are

$$\lambda_l = (\hbar^2 \frac{l^2}{a^2} + m^2)^{1/2}, \qquad l \in \mathbb{N}. \tag{15}$$

Note, that if we consider the Hamiltonian H_{FRW} of *massless* scalar particle in closed Friedmann-Robertson-Walker space-time with the metric $ds^2 = dt^2 - a(t)^2 d\Omega^2$ then because of conformal invariance of quantum Hamiltonian it is easy to see that the energy eigenvalues in this case are also given by (15) (with $m = 0$ in this case).

Now we quantize the Hamiltonian H in more general case. To simplify the calculations in the following we shall consider only the case when $g_{00} = 1$. Let $N = T^* M_0$ be reduced phase space of our system. The most general quantization formula for an appropriate function $f \in C^\infty(N)$ in geometric quantization method can be written in the following form [20]

$$\mathcal{Q}(f)\sigma = i\hbar \frac{d}{dt}(\tilde{\phi}_f^t \sigma)|_{t=0}, \qquad \sigma \in \mathcal{H}. \tag{16}$$

Here $\tilde{\phi}_f^t$ is one-parameter group of transformation of Hilbert space $\mathcal{H}$ induced by the function f via Poisson brackets. If λ_0 is nonvanishing local section of prequantization bundle $\mathcal{L}$, then $\sigma = \psi\, \lambda_0 \otimes \mu_0$. Using BKS kernel method it is possible to rewrite (16) in the form

$$\mathcal{Q}(f)[\psi\, \lambda_0 \otimes \mu_0] = i\hbar \frac{d}{dt} \psi_t|_{t=0} \lambda_0 \otimes \nu_{\tilde{\xi}}, \tag{17}$$

$$\psi_t(v^\alpha) = (i\hbar)^{-3/2} \int [\det \omega(\,\mathrm{ad}\,(v^\mu), \phi_f^t\, \mathrm{ad}\,(v^\nu))]^{1/2}$$

$$\exp[i/\hbar \int_0^t (\theta(\,\mathrm{ad}\,(f)) - f) \circ \phi_f^{-s} ds] \psi(v^\alpha \circ \phi_f^{-t}) d^n p. \tag{18}$$

where $\omega = d\theta$ and θ is the action 1-form, chosen as folows: $\theta = p_\alpha dv^\alpha$. For $f = H$ we have

$$L = \theta(\,\mathrm{ad}\,(H)) - H = \frac{-m^2}{H}. \tag{19}$$

Note, that for $m = 0$ the formula (19) indicates that the system has zero Lagrangian.

The orbits of the one-parameter group ϕ_H^t project onto geodesics in M_0

$$\frac{d^2}{dt^2}(v^\alpha \circ \phi_H^t) = -\Gamma^\alpha_{\mu\nu} \circ \phi_H^t \frac{d}{dt}(v^\mu \circ \phi_H^t) \frac{d}{dt}(v^\nu \circ \phi_H^t)$$

where $\Gamma^\alpha_{\mu\nu}$ are Christoffel symbols of the metric γ. The integrand in (18) can be simplified if the coordinates v^α are normal in a point $y_0 \in N_{\mathrm{tr}}$. In this case the functions v^α depend linearly on parameter t along the geodesics originating in y_0: $v^\alpha \circ \phi_H^t = t v^\alpha \circ \phi_H^1$ and $\gamma_{\alpha\beta} = \delta^\alpha_\beta$. In this coordinates $\Gamma^\alpha_{\mu\nu}|_{y_0} = 0$, $\partial_\alpha \gamma_{\mu\nu}|_{y_0} = 0$.

We approximate the integrand in (18) so that the integration will give results accurate to first order in t. For this, it suffices to approximate the integrand up to first

order in t and order 2 in tp^α. Equation $\frac{d}{dt}(v^\alpha \circ \phi_H^t) = \gamma^{\alpha\beta} p_\beta H^{-1}$ written in the normal coordinates yields

$$v^\alpha \circ \phi_H^{-t} = -\frac{t\gamma^{\alpha\beta} p_\beta}{H} + \text{higher order terms.} \tag{20}$$

From here we find [20], p. 132

$$\phi_H^t \,\mathrm{ad}\,(v^\alpha) = \,\mathrm{ad}\,(v^\alpha \circ \phi_H^{-t}),$$

hence,

$$\omega(\,\mathrm{ad}\,(v^\alpha), \phi_H^t \,\mathrm{ad}\,(v^\beta)) = [\,\mathrm{ad}\,(v^\alpha)(v^\beta - \frac{t\gamma^{\beta\mu} p_\mu}{H})] + \text{higher order terms} =$$

$$t\delta^\alpha_\beta H^{-3} + tp_\alpha p_\beta H^{-3} + \text{higher order terms}$$

and

$$(\det(\omega(\,\mathrm{ad}\,(v^\beta), \phi_H^t \,\mathrm{ad}\,(v^\gamma)))^{1/2} = t^{3/2} m (\sum_\sigma (p_\sigma)^2 + m^2)^{-5/4} + \text{higher order terms.} \tag{21}$$

In the following considerations we restrict ourselves only to the case $m \neq 0$ so we can omit next t-term in this expression.

Using (19), (20) and (21) we can rewrite (18) in the following form

$$\psi_t(0) = (i\hbar)^{-3/2} \int t^{3/2} m (\sum_\sigma (p_\sigma)^2 + m^2)^{-5/4} \psi(\frac{-tp_\alpha}{\sqrt{\sum (p_\sigma)^2 + m^2}}) d^3 p. \tag{22}$$

This formula defines the quantum Hamiltonian of scalar particle in curved space-time.

The Hamiltonian has the complicated form and it is hard to calculate it directly. However, it is possible consider some approximation of the Hamiltonian in order to simplify the calculations. In the next section we investigate a lower order (with respect to momenta) expansion of the Hamiltonian. Using geometric quantization approach it is in principle possible to calculate the quantum operator corresponding to each term of the lower order expansion. We calculate such the operators for the first three terms.

5 Approximation for small values of momenta

In this section an approximation of the formula (22) for small values of momenta will be given. By small momenta we mean the points of the reduced phase space T^*M_0 obeying $\gamma^{\alpha\beta} p_\alpha p_\beta << m^2$. In this approximation the Hamiltonian H takes the form

$$H = H_1 + H_2 + H_3 + \text{higher order terms.} \tag{23}$$

where

$$H_1 = m\sqrt{g_{00}}, \qquad H_2 = \sqrt{g_{00}}\, \frac{g^{\alpha\beta} p_\alpha p_\beta}{2m}, \qquad H_3 = m\sqrt{g_{00}}\, \frac{(g^{\alpha\beta} p_\alpha p_\beta)^2}{4m^3}.$$

Let us quantize the function H given by the formula (23). For simplicity we consider only the case $g_{00}=1$. Quantizing the first two terms in this expression we get

$$m-\frac{\hbar^2}{2m}g^{\alpha\beta}\nabla_\alpha\nabla_\beta+\frac{\hbar^2}{12m}R \tag{24}$$

where ∇_α is the covariant derivative with respect to 3-metric $g_{\alpha\beta}$ and R is the scalar curvature.

Let us consider the third term in (23). In normal coordinates we have $g_{\alpha\beta}(0)=\delta^\alpha_\beta$. Since $\partial_\gamma g_{\alpha\beta}(0)=0$ in normal coordinates we do not take in account higher order terms (which are always proportional to first derivatives), then, according to (17), the BKS-kernel quantization of $(g^{\alpha\beta}p_\alpha p_\beta)^2$ coincides with the quantization of the function $\sum(p_\alpha)^4$.

Let r be a real polynomial. It is easy to demonstrate [9], that the following *higher degree Von Neumann rule* follows from the general quantization axioms (see Sec. 3)

$$\mathcal{Q}(r(p))=r(\mathcal{Q}(p)).$$

From here we have

$$\mathcal{Q}\left(H_3\right)\psi(v)=\frac{\hbar^4}{4m^2}\sum_\alpha\frac{\partial^4\psi(v)}{\partial v_\alpha^4}(0). \tag{25}$$

Taking in account the equation [20]

$$\psi(v^\alpha(y))=\psi(y)|\det g(y)|^{1/4}. \tag{26}$$

we can find the expression for $(\mathcal{Q}H_3)\psi$ in an appropriate coordinates.

In order to do it we first list all possible independent invariant terms which could appear in this expression:

$$\begin{gathered}I_1=R^2\psi,\quad I_2=\triangle R\psi,\quad I_3=R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}\psi,\\ I_4=R^{\alpha\beta}R_{\alpha\beta},\quad I_5=R^\alpha\psi_{,\alpha},\quad I_6=R^{\alpha\beta}\psi_{,\alpha\beta},\\ I_7=R\triangle\psi,\quad I_8=g^{\alpha\beta}g^{\mu\nu}\psi_{,\alpha\beta\mu\nu}.\end{gathered} \tag{27}$$

Here the comma denotes the covariant derivative ∇_α, $R_{\alpha\beta\mu\nu}$, $R_{\alpha\beta}$ and R are Riemann, Ricci and scalar curvatures and $\triangle=g^{\alpha\beta}\nabla_\alpha\nabla_\beta$. Using (26), in normal coordinates we find from (25)

$$\begin{gathered}g^{\alpha\beta}g^{\mu\nu}\partial_{\alpha\beta\mu\nu}(|g|^{1/4}\psi)=g^{\alpha\beta}g^{\mu\nu}(|g|^{-3/4}\partial_{\beta\mu}|g|\partial_{\alpha\lambda}\psi+|g|^{-3/4}\partial_{\beta\lambda\mu}|g|\partial_\alpha\psi+\\ \frac{1}{2}|g|^{-3/4}\partial_{\lambda\mu}|g|\partial_{\alpha\beta}\psi+(-\frac{3}{8}|g|^{-7/4}\partial_{\alpha\lambda}|g|\partial_{\beta\mu}\psi-\frac{3}{16}|g|^{-7/4}\partial_{\alpha\beta}|g|\partial_{\lambda\mu}\psi\\ +\frac{1}{4}|g|^{-3/4}\partial_{\alpha\beta\lambda\mu}|g|)\psi+|g|^{-1/4}\partial_{\alpha\beta\lambda\mu}\psi)\end{gathered} \tag{28}$$

where $|g|=\det(g_{\alpha\beta})$.

Using the following expansion for the components of the metric g in normal coordinates [18]

$$g_{\mu\nu} = g_{\mu\nu}(0) + \frac{1}{3}R_{\mu\alpha\nu\beta}(0)v^{\alpha}v^{\beta} - \frac{1}{6}R_{\mu\alpha\nu\beta,\gamma}(0)v^{\alpha}v^{\beta}v^{\gamma} +$$
$$(\frac{1}{20}R_{\mu\alpha\nu\beta,\gamma\lambda}(0) + \frac{2}{45}R_{\alpha\mu\beta\sigma}R^{\sigma}_{\gamma\nu\lambda}(0))v^{\alpha}v^{\beta}v^{\gamma}v^{\lambda} + \text{higher order terms},$$

Using this formula, after complicated but straightforward calculations, we get

$$\mathcal{Q}\,(H_3)\;\psi\;\lambda_0 \otimes \mu_0 = |g|^{1/4}(-\frac{1}{12}I_1 + \frac{1}{5}I_2 + \frac{3}{10}I_3 + \frac{1}{30}I_4 - \frac{1}{3}I_5 + \frac{1}{3}I_7 + I_8)\lambda_0 \otimes \mu_0.$$

Using the proposed approach it is in principle possible to calculate quantum operators corresponding to all terms in the lower order expansion (23) of the curved space particle Hamiltonian. In order to do it, for example, for the term $(g^{\alpha\beta}p_{\alpha}p_{\beta})^3$ one should calculate all possible invariants of $6th$ order (the analog of (27)) in normal coordinates and, then rewrite the expression

$$g^{\alpha\beta}g^{\lambda\mu}g^{\nu\rho}\partial_{\alpha\beta\lambda\mu\nu\rho}(|g|^{1/4}\psi)$$

in a covariant form.

6 Quantization of polynomial Hamiltonians

Let M be a smooth manifold and $f \in C^{\infty}(T^*M)$ is an observable for a dynamical system with phase space T^*M. In some cases it can be usefull to expand f in neighborhood of the points $p = 0$ (i.e. for small values of momenta

$$f = f_{(0)}(q) + f_{(1)}(q)_{\alpha}p^{\alpha} + f_{(2)}(q)_{\alpha\beta}p^{\alpha}p^{\beta} + f_{(3)}(q)_{\alpha\beta\gamma}p^{\alpha}p^{\beta}p^{\gamma} + \ldots. \tag{29}$$

In order to construct Hilbert space operator for such kind of the functions we propose the following procedure of quantization of the functions polynomial in momenta which we will call *"polynomial Hamiltonians"*.

First component of this procedure is the global section $\lambda_0 \otimes \mu_0$ of the bundle $\mathcal{L} \otimes \sqrt{\wedge^n F}$ of $\mathcal{L}$-valued half-forms normal to (vertical) polarization. As it was said in Sec. 4, such section can be constructed using a non-degenerate 2-form field on the configuration space.

In fact we can choose *any* bilinear form g to construct the section $\lambda_0 \otimes \mu_0$. But in applications we usually interest in *natural choice* of g.

The problem simplifies if the form $f_{(2)}(q)_{\alpha\beta}$ in Eq. (29) is non-degenerate. In this case the section $\lambda_0 \otimes \mu_0$ can be defined using this form. However, if $f_{(2)}$ is degenerate one should choose $\lambda_0 \otimes \mu_0$ using physical considerations.

When global section of $\mathcal{L} \otimes \sqrt{\wedge^n F}$ is chosen one has to develop some approach for quantization of the terms in (29) for any order. The first two terms leave the vertical polarization invariant and so we can write down the corresponding operators

$$f = f_{(0)} + f_{(1)}(q)_{\alpha}p^{\alpha},$$

$$\mathcal{Q}(f)\psi\,\lambda_0\otimes\mu_0 = (-i\hbar\,\mathrm{ad}\,(f)\psi + (f_{(0)}(q) - \frac{i\hbar}{2}\frac{\partial f_{(1)}(q)_\alpha}{\partial q^\alpha})\psi)\lambda_0\otimes\mu_0.$$

Let us consider quantization the terms of order 2 and higher in momenta. As an example we take the function

$$f_n = f_{(n)}^{\alpha_1\alpha_2\ldots\alpha_n} p_{\alpha_1}p_{\alpha_2}\ldots p_{\alpha_n}$$

of order n.

Let us introduce in a neighborhood of a point $x\in M$ coordinates such that first derivative of $f_{(n)}^{\alpha_1\alpha_2\ldots\alpha_n}$ in x are equal to zero. We call such coordinates *normal with respect to* f_n. Then it is easy to see that the BKS-kernel quantization of f_n coincides with the quantization of the function

$$f_{(n)}^{\alpha_1\alpha_2\ldots\alpha_n}|_x p_{\alpha_1}p_{\alpha_2}\ldots p_{\alpha_n}.$$

From here we find

$$\mathcal{Q}(f_n)\psi(v) = f_{(n)}^{\alpha_1\alpha_2\ldots\alpha_n}|_x \frac{\partial^n\psi(v)}{\partial\alpha_1\partial\alpha_2\ldots\partial\alpha_n}(0).$$

Using (26), we can write

$$\mathcal{Q}(f_n)\psi(y)\,\lambda_0\otimes\mu_0 = f_{(n)}^{\alpha_1\alpha_2\ldots\alpha_n}\,\frac{\partial^n(|g|^{1/4}\psi(y))}{\partial\alpha_1\partial\alpha_2\ldots\partial\alpha_n}\,\lambda_0\otimes\mu_0.$$

Now, in order to obtain the final result one have simply to rewrite the last formula in a covariant way as it was shown in previous section.

Acknowledgments

The author is indebted to A. Aminova, I. Mikytiuk, and E. Patrin for comments, suggestions and useful discussions. The work was partially supported by Russian Foundation for Fundamental Investigation.

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